ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 2, 1990

A BANACH SPACE ADMITS A LOCALLY UNIFORMLY ROTUND NORM IF ITS DUAL IS A VAŠÁK SPACE

BY

MARIÁN FABIAN AND STANIMIR TROYANSKI

Sibeliova 49, 162 00 Prague 6, Czechoslovakia; and
Department of Mathematics and Informatics, University of Sofia, 5, Boullevard A. Ivanov, 1126 Sofia, Bulgaria

Dedicated to the memory of Zdeněk Frolík

ABSTRACT

Let V be a Banach space whose dual V^* is Vašák, that is, weakly countably determined. Then an equivalent locally uniformly rotund norm on V is constructed. According to a recent example of Mercourakis, this is a real extension of an earlier result of Godefroy, Troyanski, Whitfield and Zizler, where V^* has been a subspace of a weakly compactly generated Banach space.

There are known four renorming results on weakly compactly generated Banach spaces: *If V is weakly compactly generated, then V admits a locally uniformly rotund norm* (Troyanski [10] modulo Amir and Lindenstrauss [1]) *and its dual V^* admits a dual strictly convex norm* (Amir and Lindenstrauss [1] modulo Day [2, pp. 94–100]). *If the dual V^* is a subspace of a weakly compactly generated Banach space then V has a locally uniformly rotund norm* (Godefroy, Troyanski, Whitfield and Zizler [4]) *and V^* also has a dual locally uniformly rotund norm* (Godefroy, Troyanski, Whitfield and Zizler [3]).

Vašák in his famous paper [11] defined a more general class than the weakly compactly generated spaces are: He says that a Banach space V is *weakly countably determined*, now also known as a *Vašák space*, if there are a subspace Σ' of $\mathbf{N}^{\mathbf{N}}$, $\mathbf{N}$ means the natural numbers, and an upper semiconti-

Received July 14, 1988 and in revised form September 18, 1989

nuous, compact valued, and surjective mapping $\phi : \Sigma' \to (V, w)$, where w denotes the weak topology. In his paper Vašák also shows by adaping the method used for the weakly compactly generated spaces that *the Vašák space is renormable in the locally uniformly convex manner*. Recently Mercourakis [6] showed that *V^* is dually strictly convexifiable provided that V is a Vašák space*. In this paper we will prove one of the remaining extensions:

THEOREM 4. *If the dual V^* of a Banach space V is a Vašák space, then V admits an equivalent locally uniformly rotund norm.*

It should be noted that Mercourakis has recently constructed a dual Vašák space which is not a subspace of a weakly compactly generated space [7]. So the above theorem is a real extension of the known one [4].

For the proof of Theorem 4 we shall need both a Vašák result that a Vašák space admits a "long sequence" of nice projections [11], [5], [9] and that of Mercourakis [6, Theorem 2.5]. The main tool in the proof will be an extension of a technique from [10] in the sense that we shall replace the use of the space $c_0(\Gamma)$ by a more general space $C_1(M \times K)$, M separable metrizable and K compact, introduced by Mercourakis in his paper [6]. Recall that $f \in C_1(M \times K)$ if and only if $f \in l_\infty(M \times K)$ and f restricted to $L \times K$ lies in $c_0(L \times K)$ for any compact $L \subset M$. We shall consider $C_1(M \times K)$ with the supremum norm, under which it is a Banach space.

We shall need.

THEOREM 1 (Vašák). *Let $(V, \|\,.\,\|)$ be a Vašák space and let μ be the first ordinal of cardinality* dens V.

Then there exists on V a projectional resolution of the identity, that is, there is a "long sequence" $\{Q_\alpha : \omega \leqq \alpha \leqq \mu\}$ of linear projections on V such that $Q_\omega \equiv 0$, $Q_\mu =$ identity, *and for all $\omega < \alpha \leqq \mu$ the following hold*:

(i) $\| Q_\alpha \| = 1$,

(ii) $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\beta$ *if* $\omega \leqq \beta \leqq \alpha$,

(iii) dens $Q_\alpha \leqq$ card α, *and*

(iv) *for each $v \in V$ the mapping $\beta \mapsto Q_\beta v$ is continuous at α in the norm topology.*

The next theorem, due to Mercourakis [6], was originally formulated and proved in the language of the spaces of continuous functions. Also the proof was performed with help of a well known Gul'ko construction [5], which is rather topological. Another proof is due to Pol [8]. In what follows we present our own, perhaps simpler, and fairly function analytical proof. Let $\mathbf{N}^*$ denote the one point compactification of the discrete set $\mathbf{N}$.

THEOREM 2 (Mercourakis). *Let V be a Vašák space.*

Then there exist $\Sigma' \subset \mathbf{N}^{\mathbf{N}}$, an ordinal ν, with card $\nu =$ dens V, *and a linear bounded one to one mapping $S: V^* \to C_1(\Sigma' \times [\omega, \nu] \times \mathbf{N}^*)$, which is also weak* to pointwise continuous.*

PROOF. Write $V = \phi(\Sigma')$, where $\Sigma' \subset \mathbf{N}^{\mathbf{N}}$ and ϕ is an upper semicontinuous compact valued mapping from Σ' to (V, w). By transfinite induction over dens V and with the help of Theorem 1 we can find a new "long sequence" of linear projections $\{P_\alpha : \alpha \in [\omega, \nu]\}$, n some ordinal with card $\nu =$ dens V, which has the same properties as Q_α (maybe except (i)) and moreover with $(P_{\alpha+1} - P_\alpha)V$ separable for every $\alpha \in [\omega, \nu)$, see [2, pp. 165–166]. So there are sequences $\{h_n^\alpha : n \in \mathbf{N}\}$, $\| h_n^\alpha \| \leqq 1/n$, which are contained and are linearly dense in $(P_{\alpha+1} - P_\alpha)V$. Further, for any $\alpha \in [\omega, \nu)$ and any $n \in \mathbf{N}$ find $\sigma_n^\alpha \in \Sigma'$ such that $h_n^\alpha \in \phi(\sigma_n^\alpha)$. Finally define a mapping $S: V^* \to l_\infty(\Sigma' \times [\omega, \nu] \times \mathbf{N}^*)$ by

$$Sv^*(\sigma, \alpha, n)$$

$$= \begin{cases} \langle v^*, h_n^\alpha \rangle & \text{if } \sigma = \sigma_n^\alpha, \quad \alpha < \nu, \quad n \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases} \qquad (\sigma, \alpha, n) \in \Sigma' \times [\omega, \nu] \times \mathbf{N}^*.$$

Clearly, S is well defined (as $\| h_n^\alpha \| \leqq 1/n$), linear, bounded, and weak* to pointwise continuous. It is also injective. In fact, take any $0 \neq v^* \in V^*$. As $\bigcup\{(P_{\alpha+1} - P_\alpha)V : \omega \leqq \alpha < \nu\}$ is linearly dense in V by Theorem 1(iv), there is $\alpha \in [\omega, \nu)$ such that v^* restricted to $(P_{\alpha+1} - P_\alpha)V$ is not identically zero. Hence there is $n \in \mathbf{N}$ such that

$$\langle v^*, h_n^\alpha \rangle \neq 0.$$

It means that $Sv^*(\sigma_n^\alpha, \alpha, n) \neq 0$; $Sv^* \neq 0$.

It remains to prove that the range of S lies in $C_1(\Sigma' \times [\omega, \nu] \times \mathbf{N}^*)$. So take any $v^* \in V^*$, fix some compact $K \subset \Sigma'$ and let $\varepsilon > 0$ be given. We are to show that Sv^* restricted to $K \times [\omega, \nu] \times \mathbf{N}^*$ belongs to $c_0(K \times [\omega, \nu] \times \mathbf{N}^*)$. By contradiction, assume there exists a one to one infinite sequence $\{(\sigma_m, \alpha_m, n_m)\}$ in $K \times [\omega, \nu) \times \mathbf{N}$ such that

$$|Sv^*(\sigma_m, \alpha_m, n_m)| > \varepsilon \qquad \text{for all } m \in \mathbf{N}.$$

As $\| h_n^\alpha \| \leqq 1/n$ for all α and all n, it follows that the sequence $\{\alpha_m\}$ must contain infinitely many different elements. Let us, for simplicity, assume that $\alpha_m \neq \alpha_k$ if $m \neq k$. Note that $\sigma_m = \sigma_{n_m}^{\alpha_m}$; thus

$$|\langle v^*, h_{n_m}^{\alpha_m}\rangle| > \varepsilon.$$

But $h_{n_m}^{\alpha_m} \in \phi(\sigma_{n_m}^{\alpha_m}) = \phi(\sigma_m) \subset \phi(K)$, which is a weakly compact set owing to the properties of ϕ. Let h be a weak cluster point of the sequence $\{h_{n_m}^{\alpha_m}\}$. As $\alpha_m \neq \alpha_k$ for $m \neq k$, it follows that $(P_{\alpha+1} - P_\alpha)h = 0$ for each $\alpha \in [\omega, \nu)$. Hence by Theorem 1(iv) $h = 0$. But on the other hand we have

$$|\langle v^*, h\rangle| \geqq \liminf_{m\to\infty} |\langle v^*, h_{n_m}^{\alpha_m}\rangle| \geqq \varepsilon > 0,$$

a contradiction. Hence Sv^* lies in $C_1(\Sigma' \times [\omega, \nu] \times \mathbf{N}^*)$. ■

The next lemma is an extension (and a small correction) of [6, Lemma 4.10(b)(ii)], where A was a singleton. Let π_1 denote the canonical projection of $M \times K$ onto M.

LEMMA. *Let M be a separable metric space and K a compact space.*

Then the topology of M has a countable base $\{U_n\}$, closed under finite intersections, and such that for every $f \in C_1(M \times K)$, for every finite set $A \subset M \times K$, and for every $\Delta > 0$ there is $n \in \mathbf{N}$ such that

$$U_n \supset \pi_1(A) \quad \text{and} \quad f((U_n \setminus \pi_1(A)) \times K) \subset (-\Delta, \Delta).$$

PROOF. Let $\mathbf{U}'$ be a countable base for the topology in M. Let $\mathbf{U}$ be the smallest family of subsets in M which contains $\mathbf{U}'$ and is closed under finite unions and finite intersections. Clearly $\mathbf{U}$ is countable; write $\mathbf{U} = \{U_n\}$.

Let $f \in C_1(M \times K)$, a finite set $A \subset M \times K$ and $\Delta > 0$ be given. It is easy to find a sequence $\{n_i\} \subset \mathbf{N}$ such that $U_{n_1} \supset U_{n_2} \supset \cdots \supset \pi_1(A)$ and

$$\pi_1(A) = \bigcap_{i=1}^{\infty} \overline{U_{n_i}}.$$

Assume now, by contradiction, that for every $i \in \mathbf{N}$ there is $(\sigma_i, k_i) \in (U_{n_i} \setminus \pi_1(A)) \times K$ such that $|f(\sigma_i, k_i)| \geqq \Delta$. Then we can easily check that the sequence $\{(\sigma_i, k_i)\}$ is infinite and the set $\{(\sigma_i, k_i)\} \cup \pi_1(A) \times K$ is compact. And this contradicts the fact that f is an element of $C_1(M \times K)$. ■

The following theorem was proved in [6]. We present here a slightly different proof.

THEOREM 3 (Mercourakis). *Let M be a separable metric space and K a compact space.*

Then $C_1(M \times K)$ admits an equivalent strictly convex norm.

PROOF. Let $\{U_n\}$ be the base from Lemma. For $X \subset M \times K$ we define a pseudonorm $|.|_X$ on $C_1(M \times K)$ by

$$|f|_X^2 = \sup\left\{\sum_{n=1}^{\infty} 4^{-n} f(\tau_n)^2 \chi_X(\tau_n) : \{\tau_n\} \subset M \times K,\ \tau_n \neq \tau_m \text{ if } n \neq m\right\},$$

where $\chi_X(\tau) = 1$ if $\tau \in X$ and $\chi_X(\tau) = 0$ if $\tau \in M \times K \setminus X$. Further define for $n \in \mathbf{N}$ pseudonorms $\|.\|_n$ by

$$\|f\|_n = |f|_{U_n \times K}$$

and finally put

$$\|f\|^2 = \sum_{n=1}^{\infty} 4^{-n} \|f\|_n^2 + \|f\|_\infty^2, \qquad f \in C_1(M \times K),$$

where $\|.\|_\infty$ is the supremum norm. As $|.|_X \leqq \|.\|_\infty$, $\|.\|$ is an equivalent norm on $C_1(M \times K)$.

Let us consider an arbitrary $h \in C_1(M \times K)$ and any $\sigma \in M$. By Lemma there is a sequence $\{n_i\} \subset \mathbf{N}$ such that for all $i \in \mathbf{N}$

$$\sigma \in U_{n_i} \quad \text{and} \quad h((U_{n_i} \setminus \{\sigma\}) \times K) \subset (-1/i, 1/i).$$

Then by the triangle inequality

$$|\, \|h\|_{n_i} - |h|_{\{\sigma\}\times K} | = |\, |h|_{U_{n_i}\times K} - |h|_{\{\sigma\}\times K} | \leqq |h|_{(U_{n_i}\setminus\{\sigma\})\times K} \leqq 1/i,$$

as $|.|_X \leqq \|.\|_\infty$. Hence

$$(1) \qquad |h|_{\{\sigma\}\times K} = \lim_i \|h\|_{n_i}.$$

Now we are ready to prove that the norm $\|.\|$ is strictly convex. Let $f, g \in C_1(M \times K)$ be such that $\|f\| = \|g\| = \frac{1}{2}\|f+g\|$. From convexity we deduce that $\|f\|_n = \|g\|_n = \frac{1}{2}\|f+g\|_n$ for all $n \in \mathbf{N}$. As $\{U_n\}$ is closed under finite intersections, it is easy to verify that for any $\sigma \in M$ there is a sequence $\{n_i\} \subset \mathbf{N}$ such that (1) holds whenever h is replaced either by f, g, or $f+g$; thus

$$|f|_{\{\sigma\}\times K} = |g|_{\{\sigma\}\times K} = \tfrac{1}{2}|(f+g)|_{\{\sigma\}\times K}.$$

Now we know that the restriction $h\,|_{\{\sigma\}\times K}$ of any $h \in C_1(M \times K)$ to the set $\{\sigma\} \times K$ belongs to $c_0(\{\sigma\} \times K)$ and we can observe that $|h|_{\{\sigma\}\times K}$ is nothing else than the Day's norm of $h\,|_{\{\sigma\}\times K}$. And since the Day's norm is strictly convex [2, pp. 94–100], we conclude that

$$f\,|_{\{\sigma\}\times K} = g\,|_{\{\sigma\}\times K}.$$

This holds for any $\sigma \in M$, therefore $f = g$. ∎

Next a crucial technical assertion follows. It is an extension of a method from [10].

PROPOSITION. *Let $(V, \|\,.\,\|)$ be a Banach space, M a separable metric space and K a compact space. Assume there exist a linear bounded one to one mapping $S: V \to C_1(M \times K)$ and points $v_i^\tau \in V$, $i \in \mathbf{N}$, $\tau \in M \times K$, such that for all $0 \neq v \in V$*

$$v \in \overline{\mathrm{sp}}\{v_i^\tau : i \in \mathbf{N},\ \tau \in \mathrm{supp}\, Sv\}.$$

Then V admits an equivalent locally uniformly rotund norm.

PROOF. (Of course we think that $M \subset \mathbf{N}^{\mathbf{N}}$ and $K = [\omega, v] \times \mathbf{N}^*$.) For $v \in V$, $j \in \mathbf{N}$ and a finite set $A \subset M \times K$ we put

$$E_A^j(v) = \mathrm{dist}(v, \mathrm{sp}\{v_i^\tau : i = 1, \ldots, j,\ \tau \in A\})^2,$$

$$F_A(v) = \sum_{\tau \in A} Sv(\tau)^2.$$

Further for $v \in V$ and $j, l, m, n \in \mathbf{N}$ put

$$G_{j,l,m,n}(v) = \sup\{F_A(v) + (1/l)E_A^j(v) : A \subset U_n \times K,\ \#A = m\},$$

where the U_n are from Lemma. Finally define a norm $|.|$ on V by

$$|v|^2 = \|v\|^2 + \sum_{j,l,m,n=1}^{\infty} 2^{-j-l-m-n} G_{j,l,m,n}(v) + \|Sv\|_M^2, \qquad v \in V,$$

where $\|\,.\,\|_M$ is a strictly convex norm on $C_1(M \times K)$, see Theorem 3. Clearly, $|.|$ is an equivalent norm.

It remains to show that $|.|$ is locally uniformly convex. Since S is one to one and $\|\,.\,\|_M$ is strictly convex, $|.|$ will be strictly convex, too. Let now $0 \neq v \in V$ and $\{v_k\} \subset V$ be such that

$$2|v|^2 + 2|v_k|^2 - |v + v_k|^2 \to 0.$$

We will be done when we show that $\{v_k\}$ is relatively compact. So let $\varepsilon > 0$ be given; we are to find a finite ε-net for $\{v_k\}$. From the assumptions there is a finite set

$$\{(\tau_i, i_1), \ldots, (\tau_r, i_r)\} \subset \operatorname{supp} Sv \times \mathbf{N}$$

such that

$$\operatorname{dist}(v, \operatorname{sp}\{v_{i_1}^{\tau_1}, \ldots, v_{i_r}^{\tau_r}\}) < \varepsilon.$$

Denote

$$A_0 = \{\tau_1, \ldots, \tau_r\}, \qquad j = \max\{i_1, \ldots, i_r\}.$$

Then surely $E_{A_0}^j(v) < \varepsilon^2$. Put $\Gamma = \pi_1(A_0) \times K$. Since Γ is compact, Sv restricted to Γ lies in $c_0(\Gamma)$ and therefore there exist $\Delta > 0$ and a finite subset A of Γ, containing A_0, such that

$$\Delta + \max\{|Sv(\tau)| : \tau \in \Gamma \setminus A\} < \min\{|Sv(\tau)| : \tau \in A\}. \tag{2}$$

Note that then also

$$E_A^j(v) < \varepsilon^2.$$

Set $m = \#A$. According to Lemma we can find $n \in \mathbf{N}$ such that $U_n \supset \pi_1(A)$ and

$$Sv((U_n \setminus \pi_1(A)) \times K) \subset (-\Delta/2, \Delta/2). \tag{3}$$

Finally we choose $l \in \mathbf{N}$ such that

$$l > 4 \| v \|^2/\Delta^2.$$

For each $k \in \mathbf{N}$ find $A_k \subset U_n \times K$, $\#A_k = m$, so that

$$c_k \equiv G_{j,l,m,n}(v + v_k) - F_{A_k}(v + v_k) - (1/l)E_{A_k}^j(v + v_k) \to 0.$$

We shall show that $A_k = A$ for all large $k \in \mathbf{N}$. Suppose there is $k \in \mathbf{N}$ with $A_k \neq A$; So $A_k \setminus A \neq \varnothing$. Then we have

$$F_{A_k}(v) + 3\Delta^2/4 < F_A(v). \tag{4}$$

Indeed, if $\pi_1(A_k) \subset \pi_1(A_0)$ then (4) follows directly from (2). If there is $\tau \in A_k$ such that $\pi_1(\tau) \notin \pi_1(A_0)$, then from (3) and (2) we have

$$Sv(\tau)^2 + 3\Delta^2/4 < \Delta^2 < \min_{\sigma \in A} Sv(\sigma)^2,$$

which implies (4). Thus

$$G_{j,l,m,n}(v) \geqq F_A(v) + \frac{1}{l} E_A^j(v)$$

$$> \left(\frac{3\Delta^2}{4} - \frac{1}{l} \| v \|^2\right) + \left(F_{A_k}(v) + \frac{1}{l} E_{A_k}^j(v)\right),$$

and so from convexity

$$2^{j+l+m+n}(2|v|^2 + 2|v_k|^2 - |v + v_k|^2)$$

$$\geqq 2G_{j,l,m,n}(v) + 2G_{j,l,m,n}(v_k) - G_{j,l,m,n}(v + v_k)$$

$$> 2\left(\frac{3\Delta^2}{4} - \frac{1}{l} \| v \|^2\right) + 2F_{A_k}(v) + \frac{2}{l} E_{A_k}^j(v) + 2F_{A_k}(v_k) + \frac{2}{l} E_{A_k}^j(v_k)$$

$$- F_{A_k}(v + v_k) - \frac{1}{l} E_{A_k}^j(v + v_k) - c_k\ ,$$

$$\geqq 2\left(\frac{3\Delta^2}{4} - \frac{1}{l} \| v \|^2\right) - c_k > \Delta^2 - c_k$$

whenever $A_k \neq A$. If now $A_k \neq A$ for infinitely many $k \in \mathbf{N}$, then, when going with these k to infinity in the last inequality, we get $0 \geqq \Delta^2$, a contradiction. Hence $A_k = A$ for all $k \in \mathbf{N}$ but finitely many of them. Then from convexity again it follows that

$$2^{j+l+m+n} l (2|v|^2 + 2|v_k|^2 - |v + v_k|^2)$$

$$\geqq 2E_A^j(v) + 2E_A^j(v_k) - E_A^j(v + v_k) - lc_k$$

$$\geqq (E_A^j(v_k)^{1/2} - E_A^j(v)^{1/2})^2 - lc_k$$

for large $k \in \mathbf{N}$. Therefore

$$\lim_k \operatorname{dist}(v_k, Y)^2 = \lim_k E_A^j(v_k) = E_A^j(v) < \varepsilon^2,$$

where

$$Y = \operatorname{sp}\{v_i^\tau : i = 1, \ldots, j, \quad \tau \in A\}.$$

Note now that Y is finite dimensional and $\{v_k\}$ is bounded. Therefore $\{v_k\}$ must have a finite ε-net. ■

THEOREM 4. *Let a dual V^* be a Vašák space.*
Then V admits an equivalent locally uniformly rotund norm.

PROOF. Let $\{P_\alpha : \omega \leqq \alpha \leqq \nu\}$ be the sequence of projections on V^* constructed in the same way as at the beginning of the proof of Theorem 2 (where V^* is replaced by V). Let $S : V^{**} \to C_1(\Sigma' \times [\omega, \nu] \times \mathbf{N}^*)$ be the corresponding mapping constructed with the help of these P_α in Theorem 2. It means that S is defined by

$$Sv^{**}(\sigma, \alpha, n) = \begin{cases} \langle v^{**}, \xi_n^\alpha \rangle & \text{if } \sigma = \sigma_n^\alpha, \quad \alpha < \nu, \quad n \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases} \qquad (\sigma, \alpha, n) \in \Sigma' \times [\omega, \nu] \times \mathbf{N}^*$$

where $\sigma_n^\alpha \in \phi^{-1}(\xi_n^\alpha)$, $\xi_n^\alpha \in V^*$, $\| \xi_n^\alpha \| \leqq 1/n$, and $\{\xi_n^\alpha : n \in \mathbf{N}\}$ are linearly dense in $(P_{\alpha+1} - P_\alpha)V^*$.

We shall verify the assumptions of Proposition. Fix any $\alpha \in [\omega, \nu)$. We know that $(P_{\beta+1} - P_\beta)V^*$ are separable spaces for each $\beta \in [\omega, \nu)$ and that by Theorem 1(iv) $\bigcup\{(P_{\beta+1} - P_\beta)V^* : \omega \leqq \beta < \nu\}$ is linearly dense in V^*. Also $\langle v^{**}, v^* \rangle = 0$ whenever $v^{**} \in (P_{\alpha+1}^* - P_\alpha^*)V^{**}$ and $v^* \in (P_{\beta+1} - P_\beta)V^*$ with $\beta \neq \alpha$. Putting these facts together we can conclude that the weak* topology restricted to bounded subsets of $(P_{\alpha+1}^* - P_\alpha^*)V^{**}$ has a countable base. Now, since the image of the unit ball of V under the canonical embedding $\kappa : V \to V^{**}$ is weakly* dense in the unit ball of V^{**}, there exists a sequence $\{u_i^\alpha\} \subset V$ such that $\{\kappa(u_i^\alpha)\}$ is weakly* dense in $(P_{\alpha+1}^* - P_\alpha^*)V^{**}$. Denote

$$\Lambda(v) = \{\alpha \in [\omega, \nu) : P_{\alpha+1}^* \circ \kappa(v) \neq P_\alpha^* \circ \kappa(v)\}, \qquad 0 \neq v \in V.$$

Fix any $0 \neq v \in V$. We claim

$$v \in \overline{\mathrm{sp}}\{u_i^\alpha : i \in \mathbf{N},\ \alpha \in \Lambda(v)\}. \tag{5}$$

We shall prove it. By contradiction assume that (5) is false. Then from the Hahn–Banach theorem there is $v^* \in V^*$ such that $\langle v^*, v \rangle > 0$ and $\langle v^*, u_i^\alpha \rangle = 0$ for all $i \in \mathbf{N}$ and all $\alpha \in \Lambda(v)$. By Theorem 1(iv) we know that $P_\alpha v^* \to v^*$ in norm if $\alpha \to \nu$. It follows by transfinite induction that there is a finite set $F \subset [\omega, \nu)$ such that

$$\left\| v^* - \sum_{\alpha \in F} (P_{\alpha+1} - P_\alpha)v^* \right\| < \frac{\langle v^*, v \rangle}{2 \| v \|}.$$

Hence

$$\left| \langle v^*, v \rangle - \sum_{\alpha \in F} \langle (P_{\alpha+1} - P_\alpha)v^*, v \rangle \right| < \langle v^*, v \rangle / 2,$$

that is,

$$\left| \langle v^*, v \rangle - \sum_{\alpha \in F \cap \Lambda(v)} \langle (P^*_{\alpha+1} - P^*_\alpha)(\kappa(v)), v^* \rangle \right| < \langle v^*, v \rangle / 2.$$

And using the properties of u_i^α, we can find $i_\alpha \in \mathbf{N}$ for each $\alpha \in F$ such that

$$\left| \langle v^*, v \rangle - \sum_{\alpha \in F \cap \Lambda(v)} \langle \kappa(u^\alpha_{i_\alpha}), v^* \rangle \right| < \langle v^*, v \rangle / 2.$$

But the left hand side here is equal to $\langle v^*, v \rangle$ (> 0), a contradiction. Thus we have proved the claim (5).

Further we claim that *for every* $\alpha \in \Lambda(v)$ *there are* $\sigma \in \Sigma'$ *and* $n \in \mathbf{N}$ *such that* $(\sigma, \alpha, n) \in \operatorname{supp} S \circ \kappa(v)$. So let $\alpha \in \Lambda(v)$, that is, $P^*_{\alpha+1} \circ \kappa(v) \neq P^*_\alpha \circ \kappa(v)$. It means there is $v^* \in V^*$ such that

$$\langle (P^*_{\alpha+1} - P^*_\alpha \circ \kappa(v), v^* \rangle \neq 0, \quad \text{i.e.,} \quad \langle \kappa(v), (P_{\alpha+1} - P_\alpha) v^* \rangle \neq 0.$$

Now we recall that the sequence $\{\xi^\alpha_n : n \in \mathbf{N}\}$ is linearly dense in $(P_{\alpha+1} - P_\alpha)V^*$. Hence there is $n \in \mathbf{N}$ such that $\langle \kappa(v), \xi^\alpha_n \rangle \neq 0$. But the left hand side here is equal to $S(\kappa(v))(\sigma^\alpha_n, \alpha, n)$ by the definition of S. Thus $(\sigma^\alpha_n, \alpha, n)$ lies in $\operatorname{supp} S \circ \kappa(v)$ and the claim is proved.

Finally for $\tau = (\sigma, \alpha, n) \in \Sigma' \times [\omega, \nu) \times \mathbf{N}$ and $i \in \mathbf{N}$ put

$$v_i^\tau = u_i^\alpha.$$

Then clearly

$$v \in \overline{\operatorname{sp}}\{v_i^\tau : i \in \mathbf{N},\ \tau \in \operatorname{supp} S \circ \kappa(v)\}.$$

All the assumptions of Proposition are now verified and therefore V admits an equivalent locally uniformly rotund norm. ■

QUESTION. Does every dual Vašák space admit an equivalent dual locally uniformly rotund norm?

ACKNOWLEDGEMENTS

This work was begun while the two authors were attending the 17th Winter School on Abstract Analysis in Srní (Czechoslovakia) in January 1989. We are glad to thank the organizers, and in particular Zdeněk Frolík and David Preiss, for the warm and friendly atmosphere of this meeting.

References

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. **88** (1968), 35–46.

2. J. Diestel, *Geometry of Banach spaces*, Selected topics, Lecture Notes in Math. No. 485, Springer-Verlag, Berlin, 1975.

3. G. Godefroy, S. Troyanski, J. Whitfield and V. Zizler, *Smoothness in weakly compactly generated Banach spaces*, J. Funct. Anal. **52** (1983), 344–352.

4. G. Godefroy, S. Troyanski, J. Whitfield and V. Zizler, *Locally uniformly rotund renorming and injection into $c_0(\Gamma)$*, Can. Math. Bull. **27** (1984), 494–500.

5. S. P. Gul'ko, *On the structure of spaces of continuous functions and their complete paracompactness*, Russ. Math. Surv. **34** (1979), 36–44 = Usp. Mat. Nauk **34** (1979), 33–40.

6. S. Mercourakis, *On weakly countably determined Banach spaces*, Trans. Am. Math. Soc. **300** (1987), 307–327.

7. S. Mercourakis, *A dual weakly K-analytic Banach space is not necessarily a subspace of a weakly compactly generated Banach space*, Manuscript.

8. R. Pol, *Note on a theorem of S. Mercourakis about weakly K-analytic Banach spaces*, Comment Math. Univ. Carolinae **29** (1988), 723–730.

9. Ch. Stegall, *A proof of the theorem of Amir and Lindenstrauss*, Isr. J. Math. **68** (1989), 185–192.

10. S. Troyanski, *On locally uniformly convex and differentiable norms in certain nonseparable Banach spaces*, Studia Math. **37** (1971), 173–180.

11. L. Vašák, *On one generalization of weakly compactly generated Banach spaces*, Studia Math. **70** (1981), 11–19.

12. V. Zizler, *Locally uniformly rotund renorming and decomposition of Banach spaces*, Bull. Aust. Math. Soc. **29** (1984), 259–265.